Exercise 2.4.6

Determine the equilibrium temperature distribution for the thin circular ring of Section 2.4.2:

(a) directly from the equilibrium problem (see Section 1.4)

(b) by computing the limit as $t \to \infty$ of the time-dependent problem

---

Solution

The governing equation for the temperature u in a thin wire of length $2L$ is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -L < x < L,\ t > 0.$$

Bending the wire into a circle, welding the ends together, and assuming perfect thermal contact results in periodic boundary conditions.

$$u(-L,t) = u(L,t) \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Perfect thermal contact implies that the temperature and the heat flux are continuous at $x = \pm L$. Even if there is an initial temperature distribution $u(x,0) = f(x)$ in the ring, the temperature will eventually reach equilibrium as t gets large. The temperature can be thought to have a steady component and an unsteady component: $u(x,t) = U(x) + w(x,t)$.

$$\frac{\partial}{\partial t}[U(x) + w(x,t)] = k\frac{\partial^2}{\partial x^2}[U(x) + w(x,t)]$$

$$\frac{\partial w}{\partial t} = k\left(\frac{d^2U}{dx^2} + \frac{\partial^2 w}{\partial x^2}\right)$$

If we set

$$\frac{d^2U}{dx^2} = 0,$$

then the previous equation becomes a PDE solely for $w(x,t)$.

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2}$$

The boundary conditions for u imply the following boundary conditions for U and w.

$$u(-L,t) = u(L,t) \quad \rightarrow \quad U(-L) + w(-L,t) = U(L) + w(L,t) \quad \rightarrow \quad \begin{cases} U(-L) = U(L) \\ w(-L,t) = w(L,t) \end{cases}$$

$$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t) \quad \rightarrow \quad \frac{dU}{dx}(-L) + \frac{\partial w}{\partial x}(-L,t) = \frac{dU}{dx}(L) + \frac{\partial w}{\partial x}(L,t) \quad \rightarrow \quad \begin{cases} \frac{dU}{dx}(-L) = \frac{dU}{dx}(L) \\ \frac{\partial w}{\partial x}(-L,t) = \frac{\partial w}{\partial x}(L,t) \end{cases}$$

Part (a)

Here we will solve the equilibrium problem.

$$\frac{d^2U}{dx^2} = 0$$

Integrate both sides with respect to x.

$$\frac{dU}{dx} = C_1$$

Apply the second boundary condition to determine C_1.

$$\frac{dU}{dx}(-L) = C_1 = C_1 = \frac{dU}{dx}(L)$$

No information about C_1 is learned. Integrate both sides of the previous equation with respect to x once more.

$$U(x) = C_1 x + C_2$$

Now apply the first boundary condition.

$$U(-L) = -C_1 L + C_2 = C_1 L + C_2 = U(L)$$

We find that C_2 remains arbitrary and that $C_1 = 0$. The equilibrium temperature is thus a constant.

$$U(x) = C_2$$

To determine this constant, integrate both sides of the original PDE with respect to x from $-L$ to L.

$$\int_{-L}^{L} \frac{\partial u}{\partial t}\,dx = \int_{-L}^{L} k\frac{\partial^2 u}{\partial x^2}\,dx$$

Bring the time derivative in front of the integral on the left and evaluate the integral on the right.

$$\frac{d}{dt}\int_{-L}^{L} u(x,t)\,dx = k\frac{\partial u}{\partial x}\bigg|_{x=-L}^{x=L}$$

Note that a total derivative is used on the left because the integral in dx wipes out the x variable.

$$\frac{d}{dt}\int_{-L}^{L} u(x,t)\,dx = k\left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(-L,t)\right]$$

Because of the periodic boundary conditions, the right side is zero.

$$\frac{d}{dt}\int_{-L}^{L} u(x,t)\,dx = 0$$

Integrate both sides with respect to t.

$$\int_{-L}^{L} u(x,t)\,dx = \text{constant} \quad \Rightarrow \quad \int_{-L}^{L} u(x,0)\,dx = \int_{-L}^{L} u(x,\infty)\,dx$$

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} C_2\,dx \quad \rightarrow \quad C_2 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Part (b)

Here we will solve the time-dependent problem.

$$\begin{aligned}&\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2}, \quad -L < x < L,\ t > 0\\ &w(-L,t) = w(L,t)\\ &\frac{\partial w}{\partial x}(-L,t) = \frac{\partial w}{\partial x}(L,t)\end{aligned}$$

The initial condition for w is obtained from the one for u.

$$\begin{aligned}u(x,0) = f(x) \quad \to \quad U(x) + w(x,0) = f(x) \quad \to \quad w(x,0) &= f(x) - U(x)\\ &= f(x) - \frac{1}{2L}\int_{-L}^{L} f(r)\,dr\end{aligned}$$

Because the PDE and its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $w(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} \quad \to \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned}w(-L,t) = w(L,t) \quad &\to \quad X(-L)T(t) = X(L)T(t) \quad &\to \quad X(-L) = X(L)\\ w_x(-L,t) = w_x(L,t) \quad &\to \quad X'(-L)T(t) = X'(L)T(t) \quad &\to \quad X'(-L) = X'(L)\end{aligned}$$

Separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. (Note that the final answer would be the same if k were kept on the right. Normally constants are grouped with t.)

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned}\frac{1}{kT}\frac{dT}{dt} &= \lambda\\ \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda\end{aligned}\right\}$$

Values of λ for which there exist nontrivial solutions to the boundary problem for X are called the eigenvalues, and the solutions themselves are called the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$X'' = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \alpha x + C_4 \sinh \alpha x$$

Take a derivative of it.

$$X'(x) = \alpha(C_3 \sinh \alpha x + C_4 \cosh \alpha x)$$

Apply the boundary conditions to obtain a system of equations involving C_3 and C_4.

$$\begin{aligned} X(-L) &= C_3 \cosh \alpha L - C_4 \sinh \alpha L = C_3 \cosh \alpha L + C_4 \sinh \alpha L = X(L) \\ X'(-L) &= \alpha(-C_3 \sinh \alpha L + C_4 \cosh \alpha L) = \alpha(C_3 \sinh \alpha L + C_4 \cosh \alpha L) = X'(L) \end{aligned}$$

$$\begin{cases} -C_4 \sinh \alpha L = C_4 \sinh \alpha L \\ -C_3 \sinh \alpha L = C_3 \sinh \alpha L \end{cases}$$

No nonzero value of α gives zero, so the only way these equations are satisfied is if $C_3 = 0$ and $C_4 = 0$. The trivial solution $X(x) = 0$ is obtained, so there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$X'' = 0.$$

Integrate both sides with respect to x.

$$X' = C_5$$

Apply the second boundary condition to determine C_5.

$$X'(-L) = C_5 = C_5 = X'(L)$$

No information is learned about C_5. Integrate both sides of the previous equation with respect to x once more.

$$X(x) = C_5 x + C_6$$

Now apply the first boundary condition.

$$X(-L) = -C_5 L + C_6 = C_5 L + C_6 = X(L)$$

We find that $C_5 = 0$ and C_6 remains arbitrary.

$$X(x) = C_6$$

Because $X(x)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $X_0(x) = 1$. With $\lambda = 0$, the ODE for T becomes

$$T' = 0 \quad \Rightarrow \quad T(t) = \text{constant}.$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$X'' = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_7 \cos \beta x + C_8 \sin \beta x$$

Take a derivative of it.

$$X'(x) = \beta(-C_7 \sin \beta x + C_8 \cos \beta x)$$

Apply the boundary conditions to obtain a system of equations involving C_7 and C_8.

$$\begin{aligned} X(-L) &= C_7 \cos \beta L - C_8 \sin \beta L = C_7 \cos \beta L + C_8 \sin \beta L = X(L) \\ X'(-L) &= \beta(C_7 \sin \beta L + C_8 \cos \beta L) = \beta(-C_7 \sin \beta L + C_8 \cos \beta L) = X'(L) \end{aligned}$$

$$\begin{cases} -C_8 \sin \beta L = C_8 \sin \beta L \\ \ \ C_7 \sin \beta L = -C_7 \sin \beta L \end{cases}$$

These equations are satisfied if

$$\begin{aligned} \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

The negative eigenvalues are $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$X(x) = C_7 \cos \beta x + C_8 \sin \beta x \quad \to \quad X_n(x) = C_7 \cos \frac{n\pi x}{L} + C_8 \sin \frac{n\pi x}{L}.$$

With this value for λ, the ODE for T becomes

$$\frac{dT}{dt} = -k\frac{n^2\pi^2}{L^2}T.$$

The general solution is written in terms of the exponential function.

$$T(t) = C_9 \exp\left(-k\frac{n^2\pi^2}{L^2}t\right) \quad \to \quad T_n(t) = \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for w is a linear combination of $X(x)T(t)$ over all the eigenvalues.

$$w(x,t) = A_0 + \sum_{n=1}^{\infty} \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

Use the initial condition now to determine the constants A_0, A_n, and B_n.

$$w(x,0) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right) = f(x) - \frac{1}{2L}\int_{-L}^{L} f(r)\,dr$$

To find A_0, integrate both sides with respect to x from $-L$ to L.

$$\int_{-L}^{L}\left[A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)\right]dx = \int_{-L}^{L}\left[f(x) - \frac{1}{2L}\int_{-L}^{L} f(r)\,dr\right]dx$$

Split up the integrals and bring the constants in front.

$$A_0\int_{-L}^{L} dx + \sum_{n=1}^{\infty}\left(A_n \underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx}_{=0} + B_n \underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx}_{=0}\right) = \int_{-L}^{L} f(x)\,dx - \frac{1}{2L}\left(\int_{-L}^{L} dx\right)\int_{-L}^{L} f(r)\,dr$$

Evaluate the integrals.

$$\begin{aligned} A_0(2L) &= \int_{-L}^{L} f(x)\,dx - \frac{1}{2L}(2L)\int_{-L}^{L} f(r)\,dr \\ &= \int_{-L}^{L} f(x)\,dx - \int_{-L}^{L} f(r)\,dr \\ &= 0 \end{aligned}$$

So then

$$A_0 = 0$$

and

$$w(x,t) = \sum_{n=1}^{\infty} \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right).$$

Because of the decaying exponential function, the limit of $w(x,t)$ as $t\to\infty$ is zero. Therefore,

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}[U(x) + w(x,t)] \\ &= U(x) + \lim_{t\to\infty} w(x,t) \\ &= \frac{1}{2L}\int_{-L}^{L} f(r)\,dr + 0 \\ &= \frac{1}{2L}\int_{-L}^{L} f(r)\,dr. \end{aligned}$$

The equilibrium temperature is the average of the initial temperature distribution in the ring.